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Mean Absolute Value and Standard Deviation of the Phase of a Constant Vector Plus a Rayleigh-Distributed Vector

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The mean absolute value of the phase and the standard deviation of the phase of a constant vector plus a Rayleigh-distributed vector are determined by an evaluation of the first and second moment integrals of the probability distribution for various values of average relative intensity of the random Rayleigh-distributed component. The results of a quadrature evaluation of the integrals are tabulated over a wide range of values of average relative intensity ($k^2=0.010$ to 1,000).

1. Introduction to the Problem

Let $E_s(t)$ be a random vector whose amplitude, $|E_s(t)|$, is Rayleigh distributed in time and whose amplitude, $|E_s(t)|$, and phase angle, Arg $E_s(t)$, are independently distributed. Rice [1][1] has introduced the concept of a constant vector, E_o, plus a randomly distributed vector, $E_s(t)$, to describe electric currents or electric fields, $E(t)=E_o+E_s(t)$, subject to various natural fluctuations. Norton has presented tables of the cumulative distribution [2] of the amplitude, $|E(t)|$, and its phase [3], $\Omega=\text{Arg}\, E(t)$, with the relative intensity, $k^2=|E_r|^2/|E_o|^2$, as a parameter, where $|E_r|$ is the root mean square value of the amplitude $|E_s(t)|$. In connection with some recent studies of ionosphere roughness, Norton [4] obtained the following probability density function [2] of the phase, $p(\Omega)$, by integrating over the joint probability distribution given by Rice [1] with respect to amplitude, E, the time derivative of the amplitude, E', and the time derivative of the phase, Ω':

$$2\pi\, p(\Omega)=\{1+\sqrt{\pi}\, z \exp(z^2)\,[1+\text{erf}(z)]\} \exp(-1/k^2) \quad (1)$$

where

$$z=\frac{\cos\,\Omega}{k}. \quad (2)$$

The phase fluctuations are especially important in the performance evaluation of radio navigation and phase modulation systems since the behavior of such systems is dependent upon the behavior of the argument or phase, Ω, of the vector, $E(t)$.

The determination of the mean absolute value, $\overline{|\Omega|}$, of the phase, $\Omega=\arg E(t)$, and the corresponding standard deviation, σ_Ω, of a constant vector, plus a Rayleigh-distributed vector, $E(t)$, by an evaluation of the first and second moment integrals of the probability density function, $p(\Omega)$, for various values of the random Rayleigh-distributed component of average relative intensity, k^2, is the object of this paper.

[1] Figures in brackets indicate the literature references at the end of this paper.

[2] It is of interest to note that H. Bremmer has independently derived the expression for $p(\Omega)$ (private communication).

2. Theory

The evaluation of the mean absolute value, $\overline{|\Omega|}$, of the vector argument or "phase," Ω, and the corresponding variance, $\sigma_\Omega{}^2$, integrals,

$$\overline{|\Omega|}=2\int_0^\pi \Omega\, p(\Omega)\, d\Omega, \quad (3)$$

and

$$\sigma_\Omega{}^2=2\int_0^\pi \Omega^2 p(\Omega)\, d\Omega, \quad (4)$$

as a function of k^2 is the prime task of this paper. The probability density function, $p(\Omega)$, is symmetric about zero, $\Omega=0$, and hence the mean, $\overline{\Omega}$, is zero. The probability distribution, $p(\Omega)$, involves the error function,

$$\text{erf}\,(z)=\frac{2}{\sqrt{\pi}}\int_0^z e^{-u^2} du, \quad (5)$$

which can be evaluated by the following convergent series:

$$\text{erf}\,(z)=\frac{2}{\sqrt{\pi}}\left[z-\frac{z^3}{1!3}+\frac{z^5}{2!5}-\frac{z^7}{3!7}+\ldots\right]. \quad (6)$$

It is frequently more efficient for large values of z, ($z>1$) to use the asymptotic expansion for real values of z,

$$1-\text{erf}\,(z)=\frac{1}{z\sqrt{\pi}}\exp(-z^2)\left[1-\frac{1}{2z^2}+\frac{1\cdot 3}{(2z^2)^2}-\frac{1\cdot 3\cdot 5}{(2z^2)^3}+\frac{1\cdot 3\cdot 5\cdot 7}{(2z^2)^4}-\ldots\right]. \quad (7)$$

The mean absolute value, $\overline{|\Omega|}$, and the variance, σ_Ω^2, integrals can be represented in the theory of Gaussian quadrature [5] as a finite sum,

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$$\overline{|\Omega|}=\frac{1}{\pi}\exp\left(-1/k^2\right)\int_0^\pi \Omega p_0(\Omega)d\Omega$$

$$=\frac{1}{\pi}\exp\left(-1/k^2\right)\sum_{m=1}^{M} W_m\Omega_m p_0(\Omega_m)+\epsilon_M, \tag{8}$$

$$\sigma_\Omega^2=\frac{1}{\pi}\exp\left(-1/k^2\right)\int_0^\pi \Omega^2 p_0(\Omega)d\Omega$$

$$=\frac{1}{\pi}\exp\left(-1/k^2\right)\sum_{m=1}^{M} W_m\Omega^2{}_m p_0(\Omega_m)+\epsilon'_M, \tag{9}$$

$m=1, 2, 3 \ldots M,$

where ϵ_M and ϵ'_M are error terms which can in general be made arbitrarily small by increasing M,

$$p_0(\Omega)=2\pi\exp\left(1/k^2\right)p(\Omega), \tag{10}$$

$$\Omega_m=\frac{1}{2}[\Omega_b-\Omega_a]x_m+\frac{1}{2}[\Omega_b+\Omega_a], \tag{11}$$

and where $\Omega_b=\pi$, and $\Omega_a=0$, are the upper and lower limits of integration, respectively. The weight functions can be determined from the limits of integration,

$$W_m=\frac{1}{2}[\Omega_b-\Omega_a]H_m. \tag{12}$$

Thus, the integer, M, and the limits of integration, Ω_a, Ω_b, determine the particular values of the integrand to be calculated in the quadrature. The "universal" constants of the theory of Gaussian quadrature, H_m, x_m, can be determined for any given $f(x)$ and various M,

$$\int_{-1}^{1} f(x)dx=\sum_{m=1}^{M} H_m f(x_m). \tag{13}$$

The abscissas, x_m, are the roots of the Legendre polynomials defined by,

$$\frac{d^m}{dx^m}(x^2-1)^m-2^m m!P_m(x)=0$$

$$P_0(x)=1$$

$$P_1(x)=x$$

$$P_2(x)=\frac{3}{2}x^2-\frac{1}{2}$$

$$P_2(x)=\frac{5}{2}x^3-\frac{3}{2}x$$

$$P_4(x)=\frac{35}{8}x^4-\frac{15}{4}x^2+\frac{3}{8}$$

$$\ldots \tag{14}$$

Polynomials of higher degree are determined by use of the recursion formula,

$$(m+1)P_{m+1}(x)+mP_{m-1}(x)-(2m+1)xP_m(x)=0. \tag{15}$$

The weight coefficients, H_m, can be determined from the roots, x_m,

$$H_m=\frac{2}{(1-x_m^2)[P'_m(x_m)]^2}. \tag{16}$$

Since the available weights, H_m, and abscissas, x_m, are limited [6], ($M=48$), it is quite possible for very precise work to split each integral somewhat arbitrarily but consistent with efficiency,

$$\int_{\Omega_a}^{\Omega_b} f(\Omega)d\Omega=\int_{\Omega_a}^{\Omega^{(1)}} f(\Omega)d\Omega+\int_{\Omega^{(1)}}^{\Omega^{(2)}} f(\Omega)d\Omega$$

$$+\int_{\Omega^{(2)}}^{\Omega^{(3)}} f(\Omega)d\Omega+\ldots+\int_{\Omega^{(n-1)}}^{\Omega_b} f(\Omega)d\Omega, \tag{17}$$

where $\Omega^{(n)}\equiv\Omega_b$, and a specified number of intervals, $n=1, 2, 3 \ldots$, has been selected with limits of integration, Ω_a, $\Omega^{(1)}$, $\Omega^{(2)}$, $\Omega^{(3)}$, $\ldots \Omega_b$. Each integral is evaluated by the previously described quadrature (8, 9) with the abscissas and weights ($M=48$).

3. Computation

The first and second moment integrals, $\overline{|\Omega|}$ and $\sigma_\Omega{}^2$, were evaluated between the limits, $\Omega_a=0$, $\Omega_b=\pi$, employing the ($M=48$) Gaussian abscissas, x_m, and weights, H_m, for values of average relative intensity, k^2, between 0.01 and 1,000 of the random Rayleigh distributed component. According to the theory of Gaussian quadrature, this integration is equivalent to fitting a polynomial of degree $2m-1=95$ at 48 points, to the integrand, which points are weighted according to previously described rules (12) at the particular values of phase, $\Omega=\Omega_m$(11).

The integration was checked at values, $k^2=0.01$, 0.1, 1, 10, 100 and 1,000, by splitting the integral into two integrals, $\Omega_a=0$, $\Omega_1=\frac{\pi}{2}$, $\Omega_b=\pi$, $n=2$, (17). The maximum difference between the value of either $\overline{|\Omega|}$ or σ_Ω obtained in this way (17) and that obtained with a single integral (9) was ± 1 in the eighth significant figure. There are indications that the last two significant figures could possible be in error in certain instances. This is probably due to the electronic data processing method [3] rather than the quadrature. The results of the computation are illustrated in figure 1 and are presented in table 1[4].

[3] I.B.M. 650-407-E.D.P. machine.

[4] The integer to the right of each table entry, if present, indicates the power of the factor of 10 by which the number is multiplied, thus positioning the decimal point. For example, 5.6513905 −2 means 0.056513905. Note that for $k^2=\infty$, $\sigma_\Omega=\pi/\sqrt{3}$ and $\overline{|\Omega|}=\pi/2$.

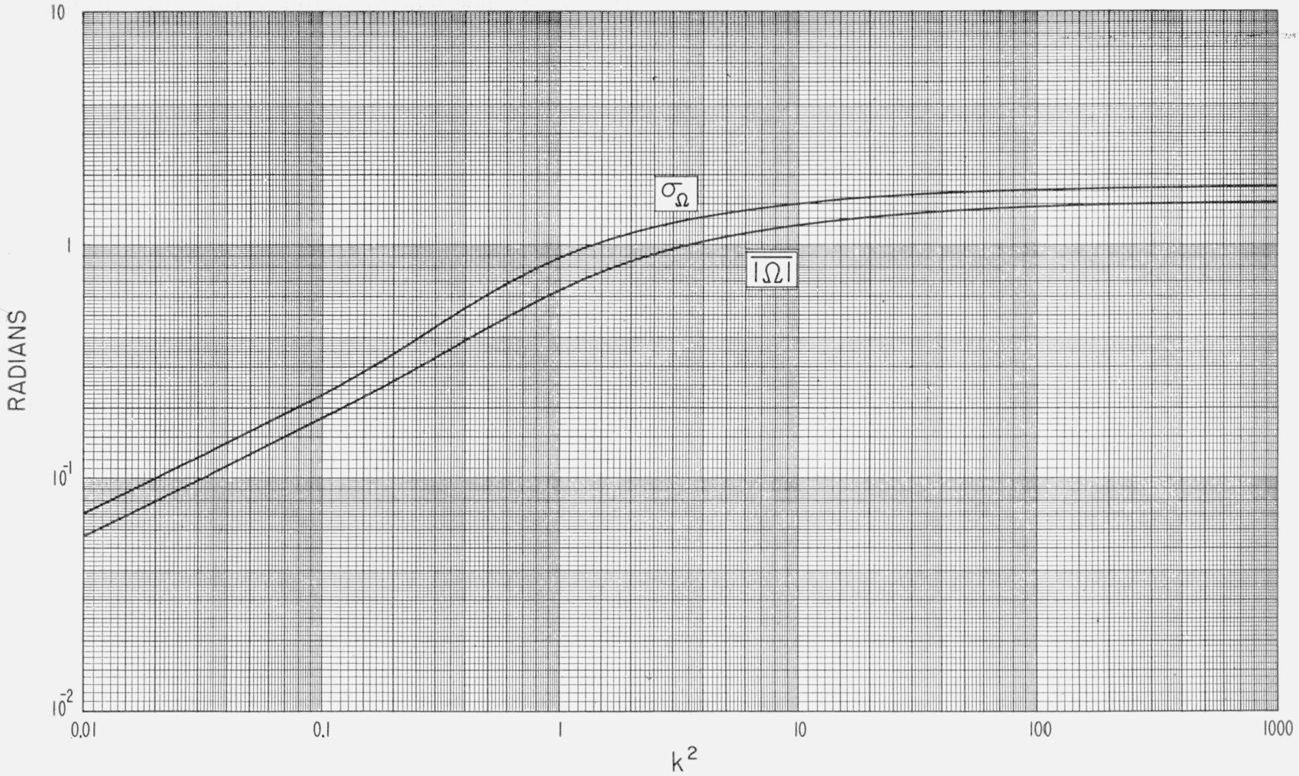

FIGURE 1. *Mean absolute value, $\overline{|\Omega|}$, and standard deviation, σ_Ω, as a function of the average relative intensity, k^2, of the random Rayleigh-distributed component.*

TABLE 1

k^2	$\overline{\lvert\Omega\rvert}$ radians	σ_Ω radians	k^2	$\overline{\lvert\Omega\rvert}$ radians	σ_Ω radians	k^2	$\overline{\lvert\Omega\rvert}$ radians	σ_Ω radians	k^2	$\overline{\lvert\Omega\rvert}$ radians	σ_Ω radians
0. 010	5. 6513905 −2	7. 0889665 −2	0. 090	1. 7204794 −1	2. 1758636 −1	0. 80	5. 7651767 −1	7. 8540860 −1	7. 0	1. 1573986	1. 4351373
. 011	5. 9281802 −2	7. 4368405 −2	. 091	1. 7303579 −1	2. 1886366 −1	. 81	5. 8018927 −1	7. 9022010 −1	7. 1	1. 1601429	1. 4378157
. 012	6. 1929049 −2	7. 7695630. −2	. 092	1. 7401916 −1	2. 2013540 −1	. 82	5. 8382359 −1	7. 9497130 −1	7. 2	1. 1628333	1. 4404390
. 015	6. 9273763 −2	8. 6933250 −2	. 095	1. 7694134 −1	2. 2391945 −1	. 85	5. 9450880 −1	8. 0887390 −1	7. 5	1. 1705980	1. 4479959
. 020	8. 0059192 −2	1. 0051253 −1	. 10	1. 8172472 −1	2. 3012770 −1	. 90	6. 1162274 −1	8. 3093795 −1	8. 0	1. 1826098	1. 4596450
. 021	8. 2050463 −2	1. 0302171 −1	. 11	1. 9099601 −1	2. 4221559 −1	. 91	6. 1494579 −1	8. 3519350 −1	8. 1	1. 1848849	1. 4618459
. 022	8. 3996053 −2	1. 0547390 −1	. 12	1. 9992282 −1	2. 5393404 −1	. 92	6. 1823668 −1	8. 3939890 −1	8. 2	1. 1871207	1. 4640069
. 025	8. 9586354 −2	1. 1252438 −1	. 15	2. 2508374 −1	2. 8749828 −1	. 95	6. 2792124 −1	8. 5172250 −1	8. 5	1. 1936018	1. 4702620
. 030	9. 8223130 −2	1. 2342886 −1	. 20	2. 6329686 −1	3. 4032320 −1	1. 0	6. 4346186 −1	8. 7133750 −1	9. 0	1. 2037103	1. 4799901
. 031	9. 9864166 −2	1. 2550275 −1	. 21	2. 7053872 −1	3. 5059207 −1	1. 1	6. 7247370 −1	9. 0744135 −1	9. 1	1. 2056364	1. 4818398
. 032	1. 0148000 −1	1. 2754525 −1	. 22	2. 7767192 −1	3. 6077869 −1	1. 2	6. 9902834 −1	9. 3992080 −1	9. 2	1. 2075323	1. 4836595
. 035	1. 0618661 −1	1. 3349834 −1	. 25	2. 9848460 −1	3. 9085106 −1	1. 5	7. 6682649 −1	1. 0205470	9. 5	1. 2130479	1. 4889462
. 040	1. 1361951 −1	1. 4291103 −1	. 30	3. 3147043 −1	4. 3923906 −1	2. 0	8. 5195625 −1	1. 1175455	10	1. 2217077	1. 4972268
. 041	1. 1505149 −1	1. 4472622 −1	. 31	3. 3783684 −1	4. 4863226 −1	2. 1	8. 6599590 −1	1. 1331300	11	1. 2372944	1. 5120705
. 042	1. 1646690 −1	1. 4652096 −1	. 32	3. 4413045 −1	4. 5792448 −1	2. 2	8. 7925345 −1	1. 1477455	12	1. 2509646	1. 5250259
. 045	1. 2061991 −1	1. 5179011 −1	. 35	3. 6258665 −1	4. 8517326 −1	2. 5	9. 1498336 −1	1. 1866651	15	1. 2836835	1. 5558026
. 050	1. 2726013 −1	1. 6022582 −1	. 40	3. 9199002 −1	5. 2841280 −1	3. 0	9. 6399295 −1	1. 2389914	20	1. 3212337	1. 5907359
. 051	1. 2854991 −1	1. 6186600 −1	. 41	3. 9767434 −1	5. 3672815 −1	3. 1	9. 7254727 −1	1. 2480052	21	1. 3271198	1. 5961754
. 052	1. 2982785 −1	1. 6349168 −1	. 42	4. 0329502 −1	5. 4493255 −1	3. 2	9. 8075157 −1	1. 2566181	22	1. 3326079	1. 6012386
. 055	1. 3359414 −1	1. 6828606 −1	. 45	4. 1978140 −1	5. 6888385 −1	3. 5	1. 0034854	1. 2803230	25	1. 3470845	1. 6145548
. 060	1. 3966386 −1	1. 7602361 −1	. 50	4. 4604899 −1	6. 0664350 −1	4. 0	1. 0361770	1. 3140093	30	1. 3662719	1. 6321174
. 061	1. 4084930 −1	1. 7753642 −1	. 51	4. 5112729 −1	6. 1388105 −1	4. 1	1. 0420645	1. 3200272	31	1. 3695492	1. 6351074
. 062	1. 4202581 −1	1. 7903826 −1	. 52	4. 5614885 −1	6. 2101680 −1	4. 2	1. 0477624	1. 3258374	32	1. 3726740	1. 6379555
. 065	1. 4550327 −1	1. 8348099 −1	. 55	4. 7088003 −1	6. 4182710 −1	4. 5	1. 0638134	1. 3421326	35	1. 3812409	1. 6457515
. 070	1. 5113924 −1	1. 9069203 −1	. 60	4. 9436251 −1	6. 7460915 −1	5. 0	1. 0875730	1. 3660625	40	1. 3933403	1. 6567297
. 071	1. 5224405 −1	1. 9210734 −1	. 61	4. 9890499 −1	6. 8089385 −1	5. 1	1. 0919370	1. 3704336	41	1. 3954936	1. 6586796
. 072	1. 5334169 −1	1. 9351423 −1	. 62	5. 0339766 −1	6. 8709150 −1	5. 2	1. 0961852	1. 3746814	42	1. 3975702	1. 6605589
. 075	1. 5659419 −1	1. 9768590 −1	. 65	5. 1658413 −1	7. 0517715 −1	5. 5	1. 1082832	1. 3867404	45	1. 4033835	1. 6658143
. 080	1. 6188698 −1	2. 0448601 −1	. 70	5. 3763008 −1	7. 3371715 −1	6. 0	1. 1265439	1. 4048374	50	1. 4118941	1. 6734927
. 081	1. 6292752 −1	2. 0582473 −1	. 71	5. 4170535 −1	7. 3919745 −1	6. 1	1. 1299436	1. 4081929	51	1. 4134456	1. 6748905
. 082	1. 6396238 −1	2. 0715680 −1	. 72	5. 4573767 −1	7. 4460535 −1	6. 2	1. 1332664	1. 4114684	52	1. 4149523	1. 6762476
. 085	1. 6703356 −1	2. 1111367 −1	. 75	5. 5758218 −1	7. 6040620 −1	6. 5	1. 1428021	1. 4208462	55	1. 4192256	1. 6800932

Table 1.—*Continued.*

k^2	$\lvert\overline{\Omega}\rvert$ radians	σ_Ω radians	k^2	$\lvert\overline{\Omega}\rvert$ radians	σ_Ω radians
60	1.4256277	1.6858461	400	1.5143646	1.7646036
61	1.4268134	1.6869104	410	1.5150558	1.7652101
62	1.4279704	1.6879488	420	1.5157220	1.7657947
65	1.4312811	1.6909181	450	1.5175857	1.7674297
70	1.4363217	1.6954337	500	1.5203113	1.7698191
71	1.4372656	1.6962787	510	1.5208079	1.7702541
72	1.4381902	1.6971060	520	1.5212898	1.7706766
75	1.4408523	1.6994873	550	1.5226565	1.7718738
80	1.4449535	1.7031527	600	1.5247021	1.7736652
81	1.4457283	1.7038446	610	1.5250810	1.7739967
82	1.4464888	1.7045237	620	1.5254503	1.7743202
85	1.4486897	1.7064882	650	1.5265072	1.7752451
90	1.4521112	1.7095402	700	1.5281154	1.7766522
91	1.4527618	1.7101201	710	1.5284164	1.7769155
92	1.4534017	1.7106904	720	1.5287113	1.7771734
95	1.4552605	1.7123466	750	1.5295601	1.7779157
100	1.4581716	1.7149389	800	1.5308671	1.7790585
110	1.4633887	1.7195798	810	1.5311138	1.7792741
120	1.4679420	1.7236250	820	1.5313563	1.7794859
150	1.4787613	1.7332182	850	1.5320572	1.7800987
200	1.4910552	1.7440872	900	1.5331466	1.7810506
210	1.4929717	1.7457784	910	1.5333538	1.7812315
220	1.4947562	1.7473525	920	1.5335573	1.7814094
250	1.4994527	1.7514919	950	1.5341489	1.7819263
300	1.5056546	1.7569509	1000	1.5350752	1.7827352
310	1.5067115	1.7578805			
320	1.5077186	1.7587659			
350	1.5104767	1.7611897			

4. Discussion and Conclusions

The problem formulated in this paper was the evaluation of two integrals—the first and second moments of the probability distribution of the phase of a constant vector plus a Rayleigh-distributed vector. These integrals were evaluated by an application of the theory of Gaussian quadrature and the results indicate that the method provides an efficient and precise solution to the problem.

It is quite possible to estimate the entire probability distribution from physical measurements of the standard deviation or the mean absolute value of the phase with the aid of the table or figure. This work therefore provides an analysis tool for the study of phase fluctuations in radio navigation and phase modulation systems.

This work was suggested by K. A. Norton and performed under National Bureau of Standards Project 8330–11–8332. The careful electronic computer work was developed by L. C. Walters.

5. References

[1] S. O. Rice, Properties of a sine wave plus random noise, Bell System Tech. J. **27**, 109 (1948). See especially eq (4.6) p. 125.

[2] K. A. Norton, L. E. Vogler, W. V. Mansfield, and P. J. Short, The probability distribution of the amplitude of a constant vector plus a Rayleigh-distributed vector, Proc. IRE **43**, 1354 (1955).

[3] K. A. Norton, E. L. Schultz, and H. Yarbrough, The probability distribution of the phase of the resultant vector sum of a constant vector plus a Rayleigh-distributed vector, J. Appl. Phys. **23**, 137 (1952). Note that k is a power ratio in this reference whereas in this paper k^2 is the power ratio and k is a voltage ratio.

[4] Private communication.

[5] Z. Kopal, Numerical Analysis, p. 367 (John Wiley & Sons, Inc., New York, N. Y., 1955). See also p. 347.

[6] For tables of H_m and x_m see P. Davis and P. Rabinowitz, Abscissas and weights for Gaussian quadrature of high order, J. Research NBS **56**, 35 (1956) RP2645. (M= 64, 80, 96 are available although apparently unpublished.)

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